

This exam contains six problems. **You should choose five problems to answer. Only five problems will be graded** if you submit six problems.

Notation: $\mathbb{D}$ denotes the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

1. Consider the polynomial

$$p(z) = z^6 + 3z^2 + 1.$$

Determine the number of zeros of $p(z)$, counted with multiplicity, in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

2. Let Ω denote the upper half of the unit disk: $\Omega = \{z \in \mathbb{C} : |z| < 1, \operatorname{Im}(z) > 0\}$. Find an explicit conformal map $f: \Omega \rightarrow \mathbb{C}^+$, $\mathbb{C}^+ = \{w \in \mathbb{C} : \operatorname{Im}(w) > 0\}$, such that

$$f(-1) = 0, \quad f(0) = 1, \quad f(1) = \infty,$$

where the values at the boundary points (i.e., on $\partial\Omega$) are understood as limits from within Ω .

3. Let $f: \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic.

- (a) Suppose $f(0) = 0$ and $f(z_0) = z_0$ for some $z_0 \in \mathbb{D} \setminus \{0\}$. Prove that $f(z) = z$ on $\mathbb{D}$.
- (b) Suppose $f(a) = a$ and $f(b) = b$ for two distinct $a, b \in \mathbb{D}$. Prove that $f(z) = z$ on $\mathbb{D}$.
- (c) Show that the conclusion of (b) fails for holomorphic $f: \mathbb{C} \rightarrow \mathbb{C}$.

4. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4} = \frac{\pi \coth(2\pi)}{4} - \frac{1}{8}.$$

Hint: Integrate $\frac{\pi \cot(\pi z)}{z^2 + 4}$ over the boundary of the square Q_N with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$ for $N \in \mathbb{Z}_{\geq 2}$. You may use the fact that one has $|\cot(\pi z)| \leq \coth(\pi/2)$ on ∂Q_N .

5. Let $f: \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic function such that for each $z \in \mathbb{D}$ we have that $|f(z)| \leq 1$. Prove that for each $z \in \mathbb{D}$ we have $|f'(z)| \leq (1 - |z|)^{-1}$.
6. Let Ω be a connected domain containing the closed unit disk, and let $f: \Omega \rightarrow \mathbb{C}$ be holomorphic. Suppose that $f(z) \neq 0$ for every $z \in \Omega$, and that $|f(z)| = 1$ whenever $|z| = 1$. Prove that f is constant on Ω .