1. Suppose X is a random variable that is independent of itself. Show that there is a constant c such that $P(X = c) = 1$.

   We have seen a few equivalent definitions of the independence of two random variables. If you plan to use one of the definitions, state it clearly. If you plan to use a property of the two random variables being independent, state it clearly.

   (If you are unable to prove the full statement, you may instead prove it for bounded X to receive partial credit.)

2. Let $(X_n)_{n\geq 1}$ be independent and identically distributed exponential random variables with rate 1. That is, the probability density function of X_1 is $f(x) = e^{-x}\mathbb{1}_{\{x>0\}}$. Prove that $S_n = \sum_{k=1}^{n} \frac{X_k}{k^2}$ converges almost surely as $n \to \infty$.

   Hint: You may either use Borel–Cantelli lemma or Kolmogorov's three-series theorem. If so then state the lemma/theorem first.

3. Let $(X_k)_{k\geq 1}$ be a sequence of independent random variables such that X_k has density

$$f_k(x) = \begin{cases} \dfrac{1}{2\sqrt{k}} & \text{if} \quad 1-\sqrt{k} \leq x \leq 1+\sqrt{k} \\ \\ 0 & \text{otherwise.} \end{cases}$$

   That is, X_k is uniformly distributed on $[1-\sqrt{k},\, 1+\sqrt{k}]$.

   Define

$$S_n = \sum_{k=1}^{n} X_k.$$

   (a) Compute $\mathbb{E}[X_k]$ and $\mathrm{Var}(X_k)$.

   (b) Find two sequences of numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ such that

$$\frac{S_n - a_n}{b_n} \Rightarrow \mathcal{N}(0,1) \qquad \text{as } n \to \infty.$$

   Here $\Rightarrow$ means convergence in distribution and $\mathcal{N}(0,1)$ means the standard normal distribution. Justify your answer rigorously.

4. Let $\alpha > 0$. Let $(X_n)_{n\geq 1}$ be independent random variables with probability mass functions

$$P(X_n = k) = \binom{n}{k} n^{-\alpha k}(1-n^{-\alpha})^{n-k}, 0 \leq k \leq n.$$

   That is, X_n is a Binomial random variable with parameters $(n, n^{-\alpha})$.

   (a) Find all the values of $\alpha > 0$ such that $X_n \to 0$ in probability but not almost surely as $n \to \infty$. Justify your answer rigorously.

   (b) Find all the values of $\alpha > 0$ such that, in addition to the requirements in (a), the sequence also satisfies that $X_n \leq 1$ eventually (i.e. $X_n > 1$ only for finitely many n's) almost surely.

   Hint: You may want to use

$$\log(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}, \quad |x| < 1.$$