Ordinary Differential Equation Preliminary Exam

May, 2026

Problem 1 Suppose A is a 4 by 4 real-valued matrix and the solutions to $|A - \lambda I| = 0$ are, counting multiplicity, $\lambda = \{-1, -1, -i, i\}$, where I is the 4 by 4 identity matrix. Write down all possible upper real Jordan canonical forms B for A. Give the general solution to $\dot{x} = Bx$ for all such upper real Jordan canonical forms B.

Problem 2

(i) Let A be a 3 constant matrix. Suppose that all solutions of

$$x' = Ax$$

are bounded as $t \to \infty$ and as $t \to -\infty$. Show that all solutions are periodic, and there is a common period for all solutions.

(ii) Let A be a 3×3 constant matrix which is skew-symmetric, meaning that the transpose $A^T = -A$. Show that all solutions are bounded as $t \to \infty$ and as $t \to -\infty$.

Problem 3 Consider the system,

$$\begin{aligned} x' &= -x^3 + y^3, \\ y' &= -x^3 - y^3 \end{aligned}$$

Prove that the origin is globally asymptotically stable.

Problem 4 Consider the nonlinear system

$$\dot{x} = -x - \frac{y}{\ln \sqrt{x^2 + y^2}}, \qquad \dot{y} = -y + \frac{x}{\ln \sqrt{x^2 + y^2}}$$

(i) Write the system in polar coordinates.

(ii) Determine if the origin is a center, a stable focus or unstable focus.

Problem 5 Consider the gradient system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = -\mathrm{grad}\, V(x, y)$$

with

$$V(x, y) = x^2(x - 1)^2 + y^2.$$

(i) Find all its equilibrium points

(ii) Classify those equilibrium points (stable node, unstable node, stable focues, unstable focus, saddle)