Real Analysis Preliminary Exam. May 4, 2026

Time allowed: 2 hours 30 minutes.

This exam contains six problems; all problems have the same weight. The worst score will be dropped; only five solutions will be counted.

Notation: $\mathbb{R}$ is the set of real numbers; $\mathbb{N}$ is the set of positive integers; m is the Lebesgue measure on $\mathbb{R}$; and m^2 is the Lebesgue measure on $\mathbb{R}^2$. Throughout the exam, "measurable" means "Lebesgue measurable".

1. Define $f\colon (0,1) \to \mathbb{R}$ by $f(x) = 1/x$. Show that if $E \subset (0,1)$ satisfies $m(E) = 0$, then $m(f(E)) = 0$. (Hint: first consider the case when $E \subset (\tau, 1)$ for some $\tau \in (0,1)$.)

2. Let $E_n \subset [0,1]$, $n \in \mathbb{N}$, be a collection of measurable sets such that $[0,1] = \cup_{n=1}^{\infty} E_n$. Suppose $f\colon [0,1] \to \mathbb{R}$ is a function such that the restriction $f|_{E_n}$ is measurable for each n. Show that f itself is measurable.

3. Let f be non-negative and integrable on $[0,1]$ and suppose that, for every $n \in \mathbb{N}$,
$$\int_{[0,1]} \big(f(x)\big)^n \, dm(x) = \int_{[0,1]} f(x) \, dm(x).$$
Show that $f = \chi_E$ a.e. on $[0,1]$, where E is a measurable subset of $[0,1]$. (Hint: consider the sets $A = \{x \in [0,1] \colon f(x) < 1\}$ and $B = \{x \in [0,1] \colon f(x) > 1\}$.)

4. Let f and $f_n, n \in \mathbb{N}$, be measurable functions on $\mathbb{R}$.
   (a) Prove that if $\lim\limits_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)|^2 \, dm(x) = 0$, then for any $\varepsilon > 0$ one has
   $$\lim_{n\to\infty} m\Big(\big\{x \in \mathbb{R} \colon |f_n(x) - f(x)| > \varepsilon\big\}\Big) = 0.$$
   (b) Prove, by providing an explicit counterexample, that the converse to the statement in part (a) is false.

5. (a) Give the definition of a function of bounded variation on $[0,1]$.
   (b) Give the definition of an absolutely continuous function on $[0,1]$.
   (c) Prove or disprove: if f is absolutely continuous on $[0,1]$, then it is of bounded variation on $[0,1]$.

6. Let T be the solid triangle with the vertices $(0,0)$, $(1,0)$, and $(1,2)$ in $\mathbb{R}^2$. Define a measure ν on $[0,1]$ as follows: $\nu(A) = m^2((A \times \mathbb{R}) \cap T)$ for measurable $A \subset [0,1]$. Prove that ν is absolutely continuous with respect to m and find the Radon–Nikodym derivative $\frac{d\nu}{dm}$.