

Statistics Prelim Exam, Stat Method part

10:00 am - 12:30 pm, Wednesday, August 19, 2026

Answer questions by showing all of your work.

1. Suppose $Y = u(X)$ is continuously distributed and the likelihood $L(\theta; x)$ has a *decreasing monotone likelihood ratio (MLR)* in the statistic $y = u(x)$; that is, for any $\theta_1 < \theta_2$, the likelihood ratio $L(\theta_1; x)/L(\theta_2; x)$ is a **monotone decreasing** function of $y = u(x)$.

- (a) (10 points) Find the most powerful (MP) size- α test for the hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1, \quad \theta_0 > \theta_1 > 0.$$

Clearly derive and write out the critical function $\phi(x)$ of the MP test.

- (b) (10 points) A test is a function $\phi(x) \in [0, 1]$, where $\phi(x)$ denotes the probability of rejecting H_0 when the observed data are x . A test has size at most α if $E_{\theta_0}[\phi(X)] \leq \alpha$. Let ϕ^* denote the MP test from part (a), and let ϕ be any other test with size at most α . Prove that

$$E_{\theta_1}[\phi^*(X)] \geq E_{\theta_1}[\phi(X)].$$

- (c) (10 points) Show that the MP test in part (a) is an unbiased test. Clearly justify your answer.
(d) (5 points) Justify that the test in part (a) is a uniformly most powerful (UMP) size- α test for the alternative hypothesis $H_1 : \theta < \theta_0$.

2. Let $X_1, \dots, X_n$ be independent and identically distributed normal random variables.

- (a) (10 points) First suppose $X_i \sim N(\mu_0, \sigma^2)$, where μ_0 is known and $\sigma^2 > 0$ is unknown. Find the UMVUE of σ^2 , and justify that it is the UMVUE.
(b) (5 points) Determine whether the estimator in part (a) attains the Cramér–Rao lower bound.
(c) (5 points) Prove directly that the estimator in part (a) converges in probability to σ^2 .
(d) (10 points) Now suppose $X_i \sim N(\mu, \sigma^2)$, where both parameters are unknown. Define

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2.$$

Prove that $\bar{X}$ and S^2 are independent. Then use their independence to identify the distribution of S^2 .

- (e) (10 points) For $n > 5$, find the UMVUE of μ/σ^4 , and justify your answer.

3. Let $X_n \mid p \sim \text{Binomial}(n, p)$, and assign p the prior $\text{Beta}(a, b)$, where $a, b > 0$. For parts (c) and (d), assume that the data are generated from $X_n \sim \text{Binomial}(n, p_0)$, where $p_0 \in (0, 1)$ is fixed.

- (a) (5 points) Find the posterior distribution of p given $X_n = x$.
(b) (5 points) Find the Bayes estimator of p under squared-error loss.
(c) (5 points) Show that the posterior mean is consistent for p_0 as $n \rightarrow \infty$.
(d) (10 points) Derive the limiting distribution of the posterior mean; specifically, find the limit in distribution of

$$\sqrt{n} \{E[p \mid X_n] - p_0\}.$$